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Termination for the Direct Sum of Left-Linear Term Rewriting Systems - Preliminary Draft*-

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1. Introduction

We prove the following conjecture [1]:

$R_0 \oplus R_1$ is left-linear and complete (complete = confluent + terminating) iff R_0 and R_1 are so.

Note that $R_0 \oplus R_1$ is confluent iff R_0 and R_1 are so [3]. Clearly, the direct sum of two systems always preserves their left-linearity. It is trivial that if $R_0 \oplus R_1$ is terminating then R_0 and R_1 are so. Thus, in this paper, we shall prove the termination property of $R_0 \oplus R_1$, assuming that R_0 and R_1 are left-linear and complete.

2. Notations and Definitions

Assuming that the reader is familiar with the basic concepts and notations concerning term rewriting systems in [3], we briefly explain notations and definitions for the following discussions.

Let F be a set of function symbols, and let V be a set of variable symbols. By $T(F, V)$, we denote the set of terms constructed from F and V .

Consider disjoint systems R_0 on $T(F_0, V)$ and R_1 on $T(F_1, V)$. Then the direct sum system $R_0 \oplus R_1$ is the term rewriting system on $T(F_0 \cup F_1, V)$. From here on the notation \rightarrow represents the reduction relation on $R_0 \oplus R_1$.

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Lemma 2.1. $R_0 \oplus R_1$ is weakly normalizing, i.e., every term M has a normal form (denoted by $M \downarrow$).

The identity of terms of $T(F_0 \cup F_1, V)$ (or syntactical equality) is denoted by \equiv . \rightarrow^* is the transitive reflexive closure of \rightarrow , \rightarrow^+ is the transitive closure of \rightarrow , \rightarrow^\equiv is the reflexive closure of \rightarrow , and $=$ is the equivalence relation generated by \rightarrow (i.e., the transitive reflexive symmetric closure of \rightarrow). \xrightarrow{m} denotes a reduction of m ($m \geq 0$) steps.

Definition. A *root* is a mapping from $T(F_0 \cup F_1, V)$ to $F_0 \cup F_1 \cup V$ as follows: For $M \in T(F_0 \cup F_1, V)$,

$$\text{root}(M) = \begin{cases} f & \text{if } M \equiv f(M_1, \dots, M_n), \\ M & \text{if } M \text{ is a constant or a variable.} \end{cases}$$

Definition. Let $M \equiv C[B_1, \dots, B_n] \in T(F_0 \cup F_1, V)$ and $C \neq \square$. Then write $M \equiv C[B_1, \dots, B_n]$ if $C[\dots]$ is a context on F_d and $\forall i, \text{root}(B_i) \in F_{\bar{d}}$ ($d \in \{0, 1\}$ and $\bar{d} = 1 - d$). Then the set $S(M)$ of the special subterms of M is inductively defined as follows:

$$S(M) = \begin{cases} \{M\} & \text{if } M \in T(F_d, V) \ (d = 0 \text{ or } 1), \\ \bigcup_i S(B_i) \cup \{M\} & \text{if } M \equiv C[B_1, \dots, B_n] \ (n > 0). \end{cases}$$

The set of the special subterms having the root symbol in F_d is denoted by $S_d(M) = \{N \mid N \in S(M) \text{ and } \text{root}(N) \in F_d\}$.

Let $M \equiv C[B_1, \dots, B_n]$ and $M \xrightarrow{A} N$ (i.e., N results from M by contracting the redex occurrence A). If the redex occurrence A occurs in some B_j , then we write $M \xrightarrow{i} N$; otherwise $M \xrightarrow{o} N$. Here, \xrightarrow{i} and \xrightarrow{o} are called an inner and an outer reduction, respectively.

Definition. For a term $M \in T(F_0 \cup F_1, V)$, the rank of layers of contexts on F_0 and F_1 in M is inductively defined as follows:

$$\text{rank}(M) = \begin{cases} 1 & \text{if } M \in T(F_d, V) \ (d = 0 \text{ or } 1), \\ \max_i \{\text{rank}(B_i)\} + 1 & \text{if } M \equiv C[B_1, \dots, B_n] \ (n > 0). \end{cases}$$

Lemma 2.2. If $M \rightarrow N$ then $\text{rank}(M) \geq \text{rank}(N)$.

Lemma 2.3. Let $M \rightarrow N$ and $\text{root}(M), \text{root}(N) \in F_d$. Then there exists a reduction $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_n \equiv N$ ($n \geq 0$) such that $\text{root}(M_i) \in F_d$ for any i .

The set of terms in the reduction graph of M is denoted by $G(M) = \{N \mid M \xrightarrow{*} N\}$. The set of terms having the root symbol in F_d is denoted by $G_d(M) = \{N \mid N \in G(M) \text{ and } \text{root}(N) \in F_d\}$.

Definition. A term M is persistent iff $G(M) = G_d(M)$ for some d .

Definition. A term M is erasable iff $M \xrightarrow{*} x$ for some $x \in V$.

From now on we assume that every term $M \in T(F_0 \cup F_1, V)$ has only x as variable occurrences, unless it is stated otherwise. Since $R_0 \oplus R_1$ is left-linear, this variable convention may be assumed in the following discussions without loss of generality. If we need fresh variable symbols not in terms, we use z, z_1, z_2, \dots .

3. Essential Subterms

In this section we introduce the concept of the essential subterms. We first prove the following property:

$$\forall N \in G_d(M) \exists P \in S_d(M), M \xrightarrow{*} P \xrightarrow{*} N.$$

Lemma 3.1. Let $M \rightarrow N$ and $Q \in S_d(N)$. Then, there exists some $P \in S_d(M)$ such that $P \equiv Q$.

R_e consists of the single rule $e(x) \triangleright x$. \xrightarrow{e} denotes the reduction relation of R_e , and $\xrightarrow{e'}$ denotes the reduction relation of $R_e \oplus (R_0 \oplus R_1)$ such that if $C[e(P)] \xrightarrow{e'} N$ then the redex occurrence Δ does not occur in P . It is easy to show the confluence property of $\xrightarrow{e'}$.

Lemma 3.2. Let $C[e(P_1), \dots, e(P_{i-1}), e(P_i), e(P_{i+1}), \dots, e(P_p)] \xrightarrow{e'}^k e(P_i)$. Then $C[P_1, \dots, P_{i-1}, e(P_i), P_{i+1}, \dots, P_p] \xrightarrow{e'}^{k'} e(P_i)$ ($k' \leq k$).

Let $M \equiv C[P] \in T(F_0 \cup F_1, V)$ be a term containing no function symbol e . Now, consider $C[e(P)]$ by replacing the occurrence P in M with $e(P)$. Assume $C[e(P)] \xrightarrow{e'}^* e(P)$. Then, by tracing the reduction path, we can also obtain the

reduction $M \equiv C[P] \xrightarrow{*} P$ (denoted by $M \xrightarrow[pull]{*} P$) under $R_0 \oplus R_1$. We say that the reduction $M \xrightarrow[pull]{*} P$ pulls up the occurrence P from M .

Example 3.1. Consider the two systems R_0 and R_1 :

$$R_0 \quad \left\{ \begin{array}{l} F(x) \rightarrow G(x, x) \\ G(C, x) \rightarrow x \end{array} \right.$$

$$R_1 \quad \left\{ \begin{array}{l} h(x) \rightarrow x \end{array} \right.$$

Then we have the reduction:

$$F(e(h(C))) \xrightarrow{e'} G(e(h(C), e(h(C)))) \xrightarrow{e'} G(h(C), e(h(C))) \xrightarrow{e'} G(C, e(h(C))) \xrightarrow{e'} e(h(C)).$$

Hence $F(h(C)) \xrightarrow[pull]{*} h(C)$. However, we cannot obtain $F(z) \xrightarrow[pull]{*} z$. Thus, in generally, we cannot obtain $C[z] \xrightarrow[pull]{*} z$ from $C[P] \xrightarrow[pull]{*} P$. \square

Lemma 3.3. Let $P \xrightarrow{*} Q$ and let $C[Q] \xrightarrow[pull]{*} Q$. Then $C[P] \xrightarrow[pull]{*} P$.

Lemma 3.4. $\forall N \in G_d(M) \exists P \in S_d(M), M \xrightarrow[pull]{*} P \xrightarrow{*} N$.

Now, we introduce the concept of the essential subterms. The set $E_d(M)$ of the essential subterms of the term $M \in T(F_0 \cup F_1, V)$ is defined as follows:

$$E_d(M) = \{P \mid P \in G(M) \cap S_d(M) \text{ and } \neg \exists Q \in G(M) \cap S_d(M) [Q \xrightarrow{\pm} P]\}.$$

The following lemmas are easily obtained from the definition of the essential subterms and Lemma 3.4.

Lemma 3.5. $\forall N \in G_d(M) \exists P \in E_d(M), P \xrightarrow{*} N$.

Lemma 3.6. $E_d(M) = \emptyset$ iff $G_d(M) = \emptyset$.

We say M is deterministic for d if $|E_d(M)| = 1$; M is nondeterministic for d if $|E_d(M)| \geq 2$. The following lemma plays an important role in the next section.

Lemma 3.7 If $\text{root}(M \downarrow) \in F_d$ then $|E_d(M)| = 1$, i.e., M is deterministic for d .

4. Termination for the Direct Sum

In this section we will show that $R_0 \oplus R_1$ is terminating. Roughly speaking, termination is proven by showing that any infinite reduction $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$ of $R_0 \oplus R_1$ can be translated into an infinite reduction $M'_0 \rightarrow M'_1 \rightarrow M'_2 \rightarrow \dots$ of R_d .

We first define the term $M^d \in T(F_d, V)$ for any term M and any d .

Definition. For any M and any d , $M^d \in T(F_d, V)$ is defined by induction on $\text{rank}(M)$:

- (1) $M^d \equiv M$ if $M \in T(F_d, V)$.
- (2) $M^d \equiv x$ if $E_d(M) = \phi$.
- (3) $M^d \equiv C[M_1^d, \dots, M_m^d]$ if $\text{root}(M) \in F_d$ and $M \equiv C[M_1, \dots, M_m]$ ($m > 0$).
- (4) $M^d \equiv P^d$ if $\text{root}(M) \in F_d$ and $E_d(M) = \{P\}$. Note that $\text{rank}(P) < \text{rank}(M)$.
- (5) $M^d \equiv C_1[C_2[\dots C_{p-1}[C_p[x]] \dots]]$ if $\text{root}(M) \in F_d$, $E_d(M) = \{P_1, \dots, P_p\}$ ($p > 1$), and every P_i^d is erasable. Here $P_i^d \equiv C_i[x] \xrightarrow[\text{pull}]{*} x$ ($i = 1, \dots, p$). Note that $\text{rank}(P_i) < \text{rank}(M)$ for any i .
- (6) $M^d \equiv x$ if $\text{root}(M) \in F_d$, $|E_d(M)| \geq 2$, and not (5).

Note that M^d is not unique if a subterm of M^d is constructed with (5) in the above definition.

Lemma 4.1. $\text{root}(M \downarrow) \notin F_d$ iff $M^d \downarrow \equiv x$.

Note. Let $E_d(M) = \{P_1, \dots, P_p\}$ ($p > 1$). Then, from Lemma 3.6 and Lemma 4.1, it follows that every P_i is erasable. Hence case (6) can be removed from the definition of M^d .

Lemma 4.2. If $P \in E_d(M)$ then $M^d \xrightarrow{*} P^d$.

We wish to translate directly an infinite reduction $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$ into an infinite reduction $M_0^d \xrightarrow{*} M_1^d \xrightarrow{*} M_2^d \xrightarrow{*} \dots$. However, the following example shows that $M_i \rightarrow M_{i+1}$ cannot be translated into $M_i^d \xrightarrow{*} M_{i+1}^d$ in generally.

Example 4.1. Consider the two systems R_0 and R_1 :

$$R_0 \quad \left\{ \begin{array}{l} F(C, x) \rightarrow x \\ F(x, C) \rightarrow x \end{array} \right.$$

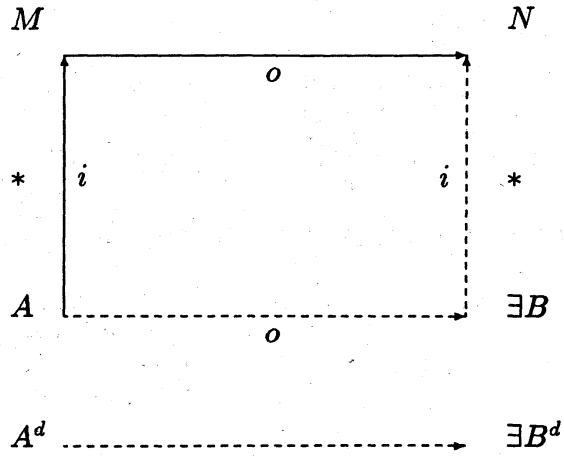
$$R_1 \quad \left\{ \begin{array}{l} f(x) \rightarrow g(x) \\ f(x) \rightarrow h(x) \\ g(x) \rightarrow x \\ h(x) \rightarrow x \end{array} \right.$$

Let $M \equiv F(f(C), h(C)) \rightarrow N \equiv F(g(C), h(C))$. Then $E_1(M) = \{f(C)\}$ and $E_1(N) = \{g(C), h(C)\}$. Thus $M^1 \equiv f(x)$, $N^1 \equiv g(h(x))$. It is obvious that $M^1 \xrightarrow{*} N^1$ does not hold. \square

Now we will consider to translate indirectly an infinite reduction of $R_0 \oplus R_1$ into an infinite reduction of R_d .

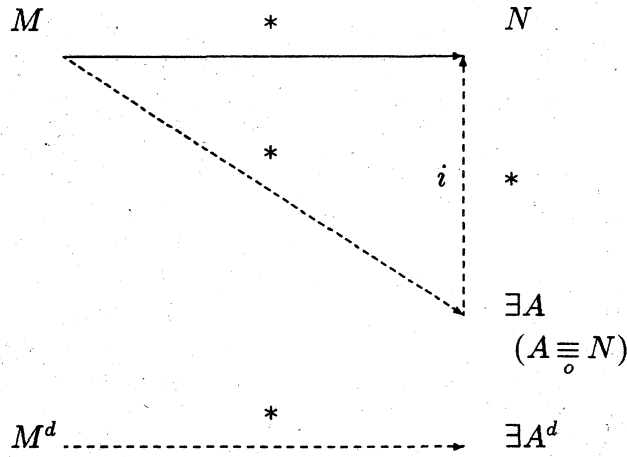
We write $M \equiv_o N$ when M and N have the same outermost-layer context, i.e., $M \equiv C[M_1, \dots, M_m]$ and $N \equiv C[N_1, \dots, N_m]$ for some M_i, N_i .

Lemma 4.3. Let $A \xrightarrow{i}^* M$, $M \xrightarrow{o} N$, $A \equiv_o M$, and $\text{root}(M), \text{root}(N) \in F_d$. Then, for any A^d there exist B and B^d such that



Proof. Let $A \equiv C[A_1, \dots, A_m]$, $M \equiv C[M_1, \dots, M_m]$, $N \equiv C'[M_{i_1}, \dots, M_{i_n}]$ ($i_j \in \{1, \dots, m\}$). Take $B \equiv C'[A_{i_1}, \dots, A_{i_n}]$. Then, we can obtain $A \xrightarrow{o} B$ and $B \xrightarrow{i} N$. From $A^d \equiv C[A_1^d, \dots, A_m^d]$ and $B^d \equiv C'[A_{i_1}^d, \dots, A_{i_n}^d]$, it follows that $A^d \rightarrow B^d$. \square

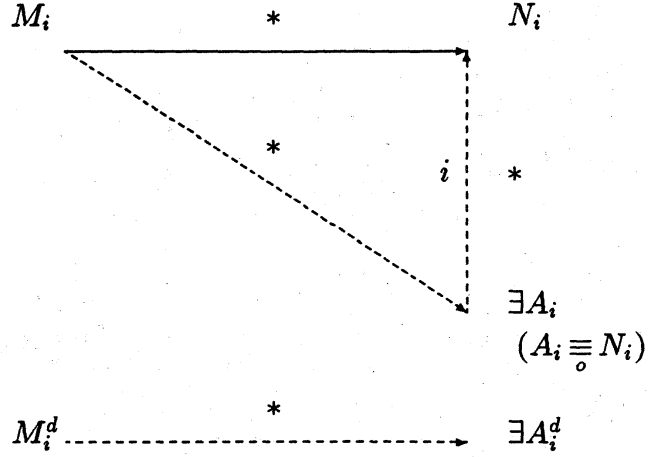
Lemma 4.4. Let $M \xrightarrow{*} N$, $\text{root}(N) \in F_d$. Then, for any M^d there exist A ($A \equiv N$) and A^d such that



Proof. We will prove the lemma by induction on $\text{rank}(M)$. The case $\text{rank}(M) = 1$ is trivial by taking $A \equiv N$. Assume the lemma for $\text{rank}(M) < k$. Then we will prove the case $\text{rank}(M) = k$. We start from the following claim.

Claim. The lemma holds if $M \xrightarrow{i}^* N$.

Proof of the Claim. Let $M \equiv C[M_1, \dots, M_m] \xrightarrow{i}^* N \equiv C[N_1, \dots, N_m]$ where $M_i \xrightarrow{i}^* N_i$ for every i . We may assume that $N_1 \equiv x, \dots, N_{p-1} \equiv x$, $\text{root}(N_i) \in F_d$ ($p \leq i \leq q-1$), and $\text{root}(N_j) \in F_{\bar{d}}$ ($q \leq j \leq m$) without loss of generality. Thus $N \equiv C[x, \dots, x, N_p, \dots, N_{q-1}, N_q, \dots, N_m]$. Then, by using the induction hypothesis, every M_i ($p \leq i \leq q-1$) has A_i ($A_i \equiv N_i$) and A_i^d such that



Now, take $A \equiv C[x, \dots, x, A_p, \dots, A_{q-1}, M_q, \dots, M_m]$. It is obvious that $M \xrightarrow{*} A$. From Lemma 2.3, we can have the reductions $A_i \xrightarrow{*} N_i$ ($p \leq i < q$) and $M_j \xrightarrow{*} N_j$ ($q \leq j \leq m$) in which every term has a root symbol in F_d . Thus it follows that $A \xrightarrow{*} N$ and $A \equiv_o N$. From Lemma 4.1 and $M_i \downarrow \equiv x$ ($1 \leq i < p$), $M_i^d \downarrow \equiv x$. Therefore, since

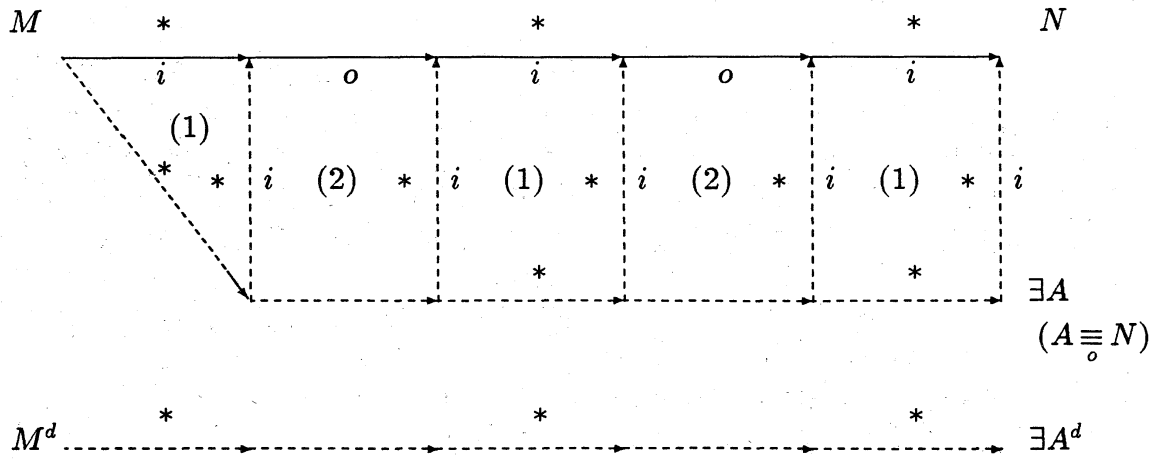
$$M^d \equiv C[M_1^d, \dots, M_{p-1}^d, M_p^d, \dots, M_{q-1}^d, M_q^d, \dots, M_m^d]$$

and $A^d \equiv C[x, \dots, x, A_p^d, \dots, A_{q-1}^d, M_q^d, \dots, M_m^d]$, it follows that $M^d \xrightarrow{*} A^d$. (end of the claim)

Now we will prove the lemma for $\text{rank}(M) = k$. Consider two cases.

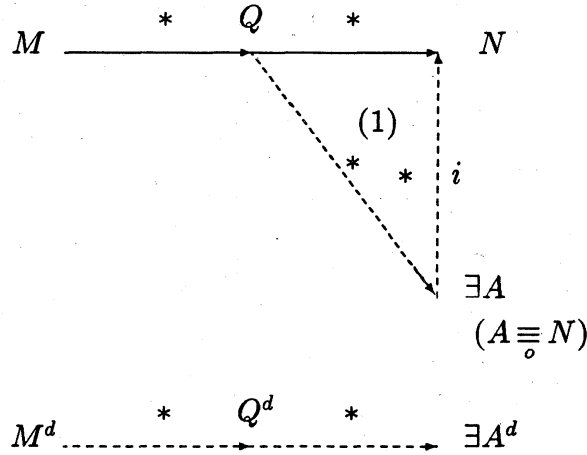
Case 1. $\text{root}(M) \in F_d$.

From Lemma 2.3, we may assume that every term in the reduction $M \xrightarrow{*} N$ has a root symbol in F_d . By splitting $M \xrightarrow{*} N$ into $M \xrightarrow{*}_i \xrightarrow{*}_o \xrightarrow{*}_i \xrightarrow{*}_o \dots \xrightarrow{*}_i N$ and using the claim for diagram (1) and Lemma 5.1 for diagram (2), we can draw the following diagram:



Case 2. $\text{root}(M) \in F_{\bar{d}}$.

Then we have some essential subterm $Q \in E_d(M)$ such that $M \xrightarrow{*} Q \xrightarrow{*} N$. From Lemma 4.2, it follows that $M^d \xrightarrow{*} Q^d$. It is obvious that $\text{rank}(Q) < k$. Hence, we can show the following diagram, drawing diagram (1) by the induction hypothesis:



□

Now we can prove the following theorem:

Theorem 4.1. Every term M has no infinite reduction.

Proof. We will prove the theorem by induction on $\text{rank}(M)$. The case $\text{rank}(M) = 1$ is trivial. Assume the theorem for $\text{rank}(M) < k$. Then, we will show the case $\text{rank}(M) = k$. Suppose M has an infinite reduction $M \rightarrow \rightarrow \rightarrow \dots$. From the induction hypothesis, we can have no infinite inner reduction $\rightarrow \rightarrow \rightarrow \dots$ in this reduction. Thus, \rightarrow_o must infinitely appear in the infinite reduction. From the induction hypothesis, all of the terms appearing in this reduction have the same rank; hence, their root symbols are in F_d if $\text{root}(M) \in F_d$. Hence, from the discussion for Case 1 in the proof of Lemma 4.4, it follows that M^d has an infinite reduction. This contradicts that R_d is terminating. \square

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